The Cascade of Circulations in Fluid Turbulence

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Kelvin's Theorem on conservation of circulations is an essential ingredient of G. I. Taylor's theory of turbulent energy dissipation by the process of vortex-line stretching. In previous work, we have proposed a nonlinear mechanism for the breakdown of Kelvin's Theorem in ideal turbulence at infinite Reynolds number. We develop here a detailed physical theory of this "cascade of circulations". Our analysis is based upon an effective equation for large-scale "coarse-grained" velocity, which contains a turbulent-induced "vortex-force" that can violate Kelvin's Theorem. We show that singularities of sufficient strength, which are observed to exist in turbulent flow, can lead to non-vanishing dissipation of circulation for an arbitrarily small filtering length in the effective equations. This result is an analogue for circulation of Onsager's theorem on energy dissipation for singular Euler solutions. The physical mechanism of the breakdown of Kelvin's Theorem is diffusion of lines of large-scale vorticity out of the advected loop. This phenomenon can be viewed as a classical analogue of the Josephson-Anderson phase-slip phenomenon in superfluids due to quantized vortex lines. We show that the circulation cascade is local in scale and use this locality to develop concrete expressions for the turbulent vortex-force by a multi-scale gradient-expansion. We discuss implications for Taylor's theory of turbulent dissipation and we point out some related cascade phenomena, in particular for magnetic-flux in magnetohydrodynamic (MHD) turbulence.

I. INTRODUCTION

The fundamental laws of vortex motion for incompressible inviscid fluids in three space dimensions were formulated by Helmholtz [1]. Starting from the incompressible Euler equations for an ideal fluid, he showed that vortex lines are material lines and that the flux within any vortex tube is a Lagrangian invariant. Lord Kelvin [2] gave an elegant alternative formulation of these laws in terms of the conservation of circulation, for any closed loop advected by an ideal fluid. This theorem is equally valid in any space dimension.

However, all of these results depend upon an implicit assumption that the solutions of the fluid equations remain smooth in the inviscid limit. In this limit, as the Reynolds number tends to infinity, all smooth, laminar solutions of the Euler equations are unstable and the fluid motion becomes turbulent. For infinite-Reynolds-number turbulent solutions, standard conservation laws of the ideal Euler equations of motion need not hold. For example, both experiments [3, 4, 5, 6] and simulations [7, 8] show that energy is not conserved in turbulent fluids even in the limit as molecular viscosity tends to zero. The anomalous rate of energy dissipation in turbulent fluids was attributed by Onsager [9] to predicted Hölder singularities in the solutions of the inviscid Euler equations. In particular, he showed that a (spatially-minimum) Hölder exponent $h_{\min} \leq 1/3$ is necessary for an Euler solution to

dissipate energy. See also [10, 11, 12, 13]. The existence of such near-singularities for turbulent velocity fields at high Reynolds number has been confirmed by data from experiments and simulations [14, 15, 16].

In a previous work [17] (hereafter referred to as "I") we considered similar questions for the conservation of circulations by turbulent solutions. In that paper we proved an analogue of Onsager's theorem, stating necessary conditions for the anomalous dissipation of circulations by inviscid Euler solutions. Furthermore, since these conditions are expected to be satisfied in turbulent flow, we conjectured that Kelvin's Theorem, in its usual form, indeed breaks down for the relevant high-Reynolds number solutions. We termed this phenomenon a "cascade of circulations." In a following paper [18] we presented evidence from direct numerical simulations for the existence of such a cascade. The purpose of the present paper is to elaborate further the physical theory of this phenomenon. In particular, our aims are as follows:

In the remainder of this section of the paper, we shall discuss some important background information. We first remind the reader of the classical Kelvin Theorem. Next we briefly review some ideas of G. I. Taylor [19, 20, 21] about the role of circulation-conservation in the production of energy dissipation in three-dimensional turbulence. In the second section of the paper we present our new results. First, we discuss the filtering approach which is the basis of our theory, and explain its relation to renormalization-group (RG) ideas and to large-eddy simulation (LES) modelling of turbulent flows. Second, we establish exact results for large-scale circulation balance of low-pass filtered velocity fields. Third, we explain how Taylor's argument can be extended to stretching of fil-

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tered vorticity and how this is related to forward cascade of energy through the inertial range. Fourth, we review the results from I on the possibility of anomalous dissipation of circulations in the limit of zero filtering length. Fifth, we point out an interesting analogy between this cascade of circulations and the phenomenon of phase-slip in superfluids, noting similarities with previous ideas of P. W. Anderson [22]. Sixth, we discuss the scale-locality of the circulation-cascade and elaborate a multi-scale gradient (MSG) expansion for circulation-flux, along the lines laid out earlier for turbulent stress [23, 24]. Finally, in the conclusion section we discuss some implications of our results and various extensions to magnetohydrodynamic (MHD) and geophysical fluid turbulence.

A. Classical Kelvin Theorem

We here briefly review some standard facts about the conservation of circulations. Let $\mathbf{u}(\mathbf{x}, t)$ be a smooth velocity field solving the incompressible Navier-Stokes equation with viscosity ν

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \nu \triangle \mathbf{u}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \quad (1)$$

where $\mathbf{x} \in \Lambda \subset \mathbf{R}^d$, for any integer $d \geq 2$. Here $p(\mathbf{x}, t)$ is the so-called "kinematic pressure" (or, thermodynamically, the enthalpy per unit mass). For any closed, oriented, rectifiable loop $C \subset \Lambda$ at an initial time t_0, one defines the circulation

$$\Gamma(C, t) = \oint_{C(t)} \mathbf{u}(t)\cdot d\mathbf{x} = \int_{S(t)} \boldsymbol{\omega}(t)\cdot d\mathbf{A} \qquad (2)$$

where $C(t)$ is the loop at time t advected by the fluid velocity, $S(t)$ is any surface spanning that loop, and $\boldsymbol{\omega}(t) = \boldsymbol{\nabla}\times\mathbf{u}(t)$ is the fluid vorticity. These circulations satisfy the Kelvin-Helmholtz theorem in the following sense:

$$\frac{d}{dt}\Gamma(C, t) = \nu \oint_{C(t)} \triangle \mathbf{u}(t)\cdot d\mathbf{x}. \qquad (3)$$

E.g., see [25], section §1.6, for the standard derivation. It is worth observing that the Kelvin theorem for all loops C is formally equivalent to the Navier-Stokes equation [26]. Indeed, if $\mathbf{u}(\mathbf{x}, t)$ is a smooth spacetime velocity field, divergence-free at all times t, then equation (3) implies that

$$\oint_C [D_t\mathbf{u}(t) - \nu \triangle \mathbf{u}(t)] \cdot d\mathbf{x} = 0 \qquad (4)$$

for all loops C at every time t. Here $D_t\mathbf{u} = \partial_t\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ is the Lagrangian time-derivative and the equation (4) is derived by applying (3) to the pre-image of the loop C at initial time t_0. By Stokes theorem, equation (4) can hold for all loops $C \subset \Lambda$ if and only if there exists a pressure-field $p(\mathbf{x}, t)$ such that the Navier-Stokes equation (1) holds locally and also globally, if the domain Λ is simply connected.

In the inviscid limit $\nu \to 0$, the circulation is formally conserved for any initial loop C. The fluid equations in this limit, the incompressible Euler equations, are the equations of motion of a classical Hamiltonian system. They can be derived by the Hamilton-Maupertuis principle from the action functional

$$S[\mathbf{x}] = \frac{1}{2}\int_{t_0}^{t_f} dt \int_\Lambda d\mathbf{a} \; |\dot{\mathbf{x}}(\mathbf{a}, t)|^2 \qquad (5)$$

with the pressure field $p(\mathbf{x}, t)$ a Lagrange multiplier to enforce the incompressibility constraint. Here $\mathbf{x}(\mathbf{a}, t)$ is the Lagrangian flow map which satisfies $\dot{\mathbf{x}}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t)$ with initial condition $\mathbf{x}(\mathbf{a}, t_0) = \mathbf{a}$. See [27, 28] for reviews. This variational principle yields the fluid equations in a Lagrangian formulation, as $\ddot{\mathbf{x}}(\mathbf{a}, t) = -\boldsymbol{\nabla}p(\mathbf{x}(\mathbf{a}, t), t)$. The Eulerian formulation (1) (with $\nu = 0$) is obtained by performing variations in the inverse map $\mathbf{a}(\mathbf{x}, t)$, or "back-to-labels map", with fixed particle positions $\mathbf{x}$. This Hamiltonian system has an infinite-dimensional gauge symmetry group consisting of all volume-preserving diffeomorphisms of Λ, which corresponds to all smooth choices of initial fluid particle labels. In this framework, the conservation of the circulations for all closed loops C emerges as a consequence of Noether's theorem for the particle-relabelling symmetry [29]. For reviews, see [27], Section 4 or [28], Section 2.2.

B. Circulation and Turbulent Energy Dissipation

In several papers [19, 20, 21], G. I. Taylor has argued for the importance of conservation of circulations in the turbulent generation of energy dissipation at high Reynolds numbers in space dimension $d = 3$. We briefly review his ideas. The simplest version of Taylor's argument is based upon the concept of vortex line-stretching. Consider a vortex tube initially with length L_0, cross-sectional area A_0, and vortex strength ω_0. Taylor assumed that such a vortex tube at high Reynolds number will evolve as a material line. Taylor also reasoned that vortex lines (or any material lines) should tend to lengthen, on average, under random advection by a turbulent velocity field. Thus, at a later time $t > t_0$, the tube length is typically $L(t) > L_0$. By incompressibility, the volume $V(t) = L(t)A(t)$ does not change in time, so that $A(t) < A_0$. Furthermore, Taylor reasoned by the Helmholtz theorem that the vortex-flux through the tube, $\Gamma(t) = \omega(t)A(t)$, would not change, so that $\omega(t) > \omega_0$. In fact, by this chain of reasoning,

$$\omega(t)/\omega_0 = L(t)/L_0 \qquad (6)$$

and vortex strength increases in direct proportion to line-length. Because the viscous energy dissipation in the vortex-tube is given by $\nu \int \omega^2(t)\, dV = \nu\omega^2(t)V_0$, this process should lead to a dramatic enhancement of dissipation.

However, this argument contains an apparent inconsistency. On the one hand, Taylor's assumptions that vortex-lines are material lines and that the Kelvin Theorem applies require that the viscosity term in the circulation balance (3) can be neglected. On the other hand, Taylor retains the viscous dissipation in the energy balance, arguing, in fact, that it is sizable. It is not at all clear that it is valid to ignore the viscosity effects in one place and to keep them in another. Taylor himself recognized the delicacy of his argument. In [20] he presented this line of reasoning, and then wrote: "When $\overline{\omega^2}$ has increased to some value which depends on the viscosity, it is no longer possible to neglect the effect of viscosity in the equation for the conservation of circulation, so that (10) [our (6)] ceases to be true." Thus, Taylor assumed that there is some interval of time or some range of length-scales for which viscous effects can be neglected in the circulation balance (3). We shall critically review this assumption below.

In a following paper [21], Taylor tested some predictions of his argument using experimental data for decaying turbulence generated from a wind-tunnel. His analysis was based upon the following equation for production of enstrophy,

$$\partial_t \left(\frac{1}{2}|\boldsymbol{\omega}|^2 \right) + \boldsymbol{\nabla}\cdot\left[\frac{1}{2}|\boldsymbol{\omega}|^2\mathbf{u} - \nu\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\omega}|^2 \right) \right]$$
$$= \boldsymbol{\omega}^\top\mathbf{S}\boldsymbol{\omega} - \nu|\boldsymbol{\nabla}\boldsymbol{\omega}|^2 \qquad (7)$$

which is an exact consequence of the incompressible Navier-Stokes dynamics (1) for space-dimension $d = 3$. Here $S_{ij} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the strain matrix. Under conditions of space-homogeneity, the average of the transport term vanishes, so that

$$(d/dt)\langle \frac{1}{2}|\boldsymbol{\omega}|^2 \rangle = \langle \boldsymbol{\omega}^\top\mathbf{S}\boldsymbol{\omega} \rangle - \nu\langle|\boldsymbol{\nabla}\boldsymbol{\omega}|^2 \rangle \qquad (8)$$

Taylor's argument on vortex-line stretching suggests that $(d/dt)\langle|\boldsymbol{\omega}|^2\rangle > 0$, which can hold if and only if $\langle\boldsymbol{\omega}^\top\mathbf{S}\boldsymbol{\omega}\rangle > \nu\langle|\boldsymbol{\nabla}\boldsymbol{\omega}|^2\rangle \geq 0$ Thus, enstrophy will be created when the mean rate of vortex-stretching by the strain is positive and exceeds the mean destruction of enstrophy by viscosity. In [21], Taylor found from an analysis of wind-tunnel data that the latter condition holds for an initial range of time in decaying turbulence.

II. CIRCULATION CASCADE

We now turn to an analysis of circulation conservation in high-Reynolds-number turbulent flow. One approach would be to directly analyze the $\nu \to 0$ limit of eq. (3). However, we shall pursue a complementary approach based upon a study of nonlinear transfer in the inertial range.

A. Filtering Approach

To analyze the dynamics in the inertial range, we introduce effective equations that govern the evolution of the velocity field at large length-scales. For any chosen length ℓ, let

$$\overline{\mathbf{u}}_\ell(\mathbf{x}) = \int d\mathbf{r}\, G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x} + \mathbf{r}) \qquad (9)$$

denote the low-pass filtered velocity at scale ℓ, where $G_\ell(\mathbf{r}) = \ell^{-d}G(\mathbf{r}/\ell)$ is a filter kernel. We shall assume that G is positive, smooth, rapidly decaying in space and with unit integral. Then $\overline{\mathbf{u}}_\ell$ satisfies an effective equation:

$$\partial_t\overline{\mathbf{u}}_\ell + (\overline{\mathbf{u}}_\ell\cdot\boldsymbol{\nabla})\overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_\ell = -\boldsymbol{\nabla}\overline{p}_\ell + \nu\triangle\overline{\mathbf{u}}_\ell, \qquad (10)$$

where $\overline{p}_\ell$ is the filtered pressure and $\boldsymbol{\tau}_\ell$ is the turbulent stress-tensor

$$\boldsymbol{\tau}_\ell = \overline{(\mathbf{u}\,\mathbf{u})}_\ell - \overline{\mathbf{u}}_\ell\overline{\mathbf{u}}_\ell. \qquad (11)$$

The filtering operation that we have employed can be regarded as a "coarse-graining" that eliminates high-wavenumber modes, as in renormalization-group methodology [30, 31]. Because of momentum conservation, the effective renormalized equation can change only by additional contributions to the stress tensor. This filtering approach is also the mathematical basis of the large-eddy-simulation (LES) modeling scheme [32, 33]. In this scheme, the stress tensor is the main unknown which must be modelled, in order to obtain a closed equation for computation of the large-scale velocity field.

In the inertial-range of turbulent flow the final viscosity term in eq.(10) can be neglected. For example, a fairly crude estimate based upon the identity

$$\triangle\overline{\mathbf{u}}_\ell(\mathbf{x}) = \ell^{-2}\int d\mathbf{r}\,(\triangle G)_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r}) \qquad (12)$$

is $\|\nu\triangle\overline{\mathbf{u}}_\ell\|_2 \leq (\nu/\ell^2)(\text{const.})\|\mathbf{u}\|_2$, where $\|\mathbf{u}\|_2 = [\int d\mathbf{x}\,|\mathbf{u}(\mathbf{x})|^2]^{1/2}$ is the L^2-norm. If the total kinetic energy per mass $E = (1/2)\|\mathbf{u}\|_2^2$ remains finite in the limit as $\nu \to 0$, then the viscosity term in eq.(10) tends to zero in L^2-norm for any fixed filter-length ℓ.

There is another form of the effective equation (10) which is useful. Note that the stress appears only via the *turbulent (subgrid) force* $\mathbf{f}_\ell = -\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_\ell$ [33]. This can be replaced in (10) using the following elementary identity

$$\mathbf{f}_\ell = -\boldsymbol{\nabla}k_\ell + \mathbf{f}_\ell^*, \qquad (13)$$

where $k_\ell = (1/2)\text{tr}\,\boldsymbol{\tau}_\ell$ is the *turbulent kinetic energy* [32] and

$$\mathbf{f}_\ell^* = \overline{(\mathbf{u}\times\boldsymbol{\omega})}_\ell - \overline{\mathbf{u}}_\ell\times\overline{\boldsymbol{\omega}}_\ell \qquad (14)$$

is the *turbulent vortex force*. With this replacement, (10) becomes

$$\partial_t\overline{\mathbf{u}}_\ell + (\overline{\mathbf{u}}_\ell\cdot\boldsymbol{\nabla})\overline{\mathbf{u}}_\ell = -\boldsymbol{\nabla}\overline{p}_\ell^* + \mathbf{f}_\ell^* + \nu\triangle\overline{\mathbf{u}}_\ell, \qquad (15)$$

where $\overline{p}_\ell^* = \overline{p}_\ell + k_\ell$ is a modified pressure. Although this form of the large-scale effective equation leads to more intuitive results, it is less easy to make sense of mathematically. In fact, the vortex force $\mathbf{f}_\ell^*$ could be badly ultraviolet divergent in the limit as $\nu \to 0$. Notice that for infinite-Reynolds-number turbulence the velocity $\mathbf{u}$ is believed to be a continuous but non-differentiable function, so that the vorticity $\boldsymbol{\omega}$ exists only as a distribution. Therefore, the product $\mathbf{u} \times \boldsymbol{\omega}$ is *a priori* ill-defined. However, the vortex force remains well-defined due to the identity (13), since both $\mathbf{f}_\ell$ and $\boldsymbol{\nabla} k_\ell$ make sense as long as $\|\mathbf{u}\|_2 < \infty$.

B. Circulation-Balance in the Large-Scales

It is natural to inquire about the circulation-balance for the large-scale effective equation. Let us choose an oriented, rectifiable, closed loop C in space. We define $\overline{C}_\ell(t)$ as the loop C advected by the filtered velocity $\overline{\mathbf{u}}_\ell$. This definition makes sense, since the filtered velocity $\overline{\mathbf{u}}_\ell$ is Lipschitz in space, and corresponding flow maps $\overline{\mathbf{x}}_\ell(\mathbf{a}, t)$ defined by

$$(d/dt)\overline{\mathbf{x}}_\ell(\mathbf{a}, t) = \overline{\mathbf{u}}_\ell(\overline{\mathbf{x}}_\ell(\mathbf{a}, t), t), \quad \overline{\mathbf{x}}_\ell(\mathbf{a}, t_0) = \mathbf{a}, \quad (16)$$

both exist and are unique (see I). We define a "large-scale circulation" with initial loop C as the line-integral

$$\overline{\Gamma}_\ell(C, t) = \oint_{\overline{C}_\ell(t)} \overline{\mathbf{u}}_\ell(t) \cdot d\mathbf{x}. \quad (17)$$

for $\ell < R =$ the radius of gyration of the loop C [71]. The same calculation that establishes the Kelvin theorem, but using the effective eq. (10) rather than Navier-Stokes eq.(1), gives

$$(d/dt)\overline{\Gamma}_\ell(C, t) = \oint_{\overline{C}_\ell(t)} [\mathbf{f}_\ell(t) + \nu \triangle \overline{\mathbf{u}}_\ell] \cdot d\mathbf{x} \quad (18)$$

If the Navier-Stokes eq.(1) were driven by an external body-force $\mathbf{f}^{\text{ext}}$, then there would be an additional term $\overline{\mathbf{f}}^{\text{ext}}$ inside the square bracket in eq.(18). If this external force is spectrally supported at wavenumbers of order $1/L$, then its contribution to the circulation balance is $O(R/L)$. Thus, the forcing term is negligible for $R \ll L$. Likewise, the viscous term in eq.(18) is negligible for small viscosity ν and fixed filter-length ℓ, by an elaboration of the argument given around eq.(12). (A so-called "trace theorem" can be used to estimate the restriction of $\triangle \overline{\mathbf{u}}_\ell$ to the loop C; see I and [34]).

These remarks show that the nonlinear term from the subgrid force is the dominant term in the circulation balance (18) for inertial-range values $L \gg R > \ell \gg \eta_d$ (where η_d is a dissipation length-scale determined by the viscosity ν). If we imagine that the total circulation at all scales on the loop is conserved, then the line-integral of $\mathbf{f}_\ell$ on the RHS of (18) represents a "transfer" of circulation to subgrid modes at length-scales $< \ell$. This motivates

the definition, for any loop C and filter length ℓ, of a *flux of circulation*

$$K_\ell(C, t) = -\oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell(t) \cdot d\mathbf{x} = -\oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell^*(t) \cdot d\mathbf{x} \quad (19)$$

so that $(d/dt)\overline{\Gamma}_\ell(C, t) = -K_\ell(C, t)$ (up to small corrections from external forcing and viscosity). We have used identity (13) to justify the equality of the two expressions in the definition (19). The minus sign has been introduced so that the signs of the circulation (17) and the circulation-flux (19) should be positively correlated. This expectation will be discussed more below.

The "circulation-flux" defined in (19) has the physical dimensions of work or of torque (per unit mass). Additional insight into its meaning can be obtained by decomposing the turbulent vortex-force (14) into components perpendicular and parallel to large-scale vortex-lines:

$$\mathbf{f}_{\perp \ell}^* = \boldsymbol{\sigma}_\ell \times \hat{\boldsymbol{\omega}}_\ell, \quad \mathbf{f}_{\parallel \ell}^* = (\mathbf{f}_\ell^* \cdot \hat{\boldsymbol{\omega}}_\ell)\hat{\boldsymbol{\omega}}_\ell \quad (20)$$

where $\hat{\boldsymbol{\omega}}_\ell = \overline{\boldsymbol{\omega}}_\ell / |\overline{\boldsymbol{\omega}}_\ell|$ and

$$\boldsymbol{\sigma}_\ell = \hat{\boldsymbol{\omega}}_\ell \times \mathbf{f}_\ell^*. \quad (21)$$

If $\hat{\mathbf{t}}_\ell$ is the unit tangent vector to the curve $\overline{C}_\ell(t)$ and s is the arc-length parameter, then

$$K_\ell(C, t) = \oint_{\overline{C}_\ell(t)} \boldsymbol{\sigma}_\ell(t) \cdot \mathbf{n}_\ell \, ds - \oint_{\overline{C}_\ell(t)} \mathbf{f}_{\parallel \ell}^*(t) \cdot d\mathbf{x} \quad (22)$$

where $\mathbf{n}_\ell = \hat{\mathbf{t}}_\ell \times \hat{\boldsymbol{\omega}}_\ell$. Note that the latter vector is normal both to lines of large-scale vorticity $\overline{\boldsymbol{\omega}}_\ell$ and to the loop $\overline{C}_\ell(t)$, but it is not generally a unit vector. The first term in (22) can be interpreted as a lateral diffusion of vortex-lines out of the advected loop, where $\boldsymbol{\sigma}_\ell$ plays the role of a transport vector of vortex-lines. The second term in (22) represents an additional work (or torque) due to the parallel component of the turbulent vortex-force.

Some particular cases of (22) are of special interest. For example, consider the case that $\overline{C}_\ell(t)$ is instantaneously a closed vortex line. (This property will not generally be preserved in time). Then the first term in (22) vanishes and $K_\ell(C, t) = -\oint_{\overline{C}_\ell(t)} \mathbf{f}_{\parallel \ell}^*(t) \cdot d\mathbf{x}$. Such integrals play an important role in vortex-reconnection theory [35]. The distinguished vortex lines for which this integral is extremal drive the reconnection process and the value of the integral for such lines gives the rate of reconnection of vortex-flux. This integral is therefore the proper point of departure for a theory of turbulent reconnection of large-scale vortex-lines. Another special case of (22) of interest is when the loop $\overline{C}_\ell(t)$ lies in a transversal surface normal to the lines of large-scale vorticity. In that case, the second term in (22) vanishes and $K_\ell(C, t) = \oint_{\overline{C}_\ell(t)} \boldsymbol{\sigma}_\ell(t) \cdot \hat{\mathbf{n}}_\ell \, ds$, where $\hat{\mathbf{n}}_\ell = \hat{\mathbf{t}}_\ell \times \hat{\boldsymbol{\omega}}_\ell$ is now a unit vector. This condition is always satisfied for space dimension $d = 2$. The flux of circulation is then entirely due to the diffusion of vortex-lines out of the loop.

These remarks on physical interpretation of $K_\ell(C, t)$ lead to some natural guesses on the correlation of its

sign with that of the circulation $\overline{\Gamma}_\ell(C,t)$. The latter can be written as

$$\overline{\Gamma}_\ell(C,t) = \int_{\overline{S}_\ell(t)} \overline{\boldsymbol{\omega}}_\ell(t)\cdot d\mathbf{A}, \qquad (23)$$

where $\overline{S}_\ell(t)$ is any smooth surface spanning the loop $\overline{C}_\ell(t)$ and with orientation consistent to that of $\overline{C}_\ell(t)$ (by the righthand rule). If the circulation (23) is positive, then there is a net contribution from vortex lines threading the loop in the direction of the surface unit normal. If the effect of the turbulence is "diffusive" on average, then one would expect that the vortex-force will tend to smooth out the excess of positive-sign vorticity threading the loop. Thus, according to the sign convention of the definition (19), we can expect that $K_\ell(C,t)$ will also tend to be positive and to reduce the overall magnitude of the large-scale circulation. Of course, this argument works equally well when $\overline{\Gamma}_\ell(C,t)$ has negative sign. We may therefore expect that there is in general a "forward cascade" of circulations, and that the magnitude of the large-scale circulation, of whatever sign, will tend to be decreased by the small-scale turbulence. This reasonable result has been confirmed by numerical results in [18].

An interesting exception is the inverse-energy cascade for $d = 2$ turbulence. For space dimension $d = 2$, the enstrophy $\Omega(t) = (1/2)|\omega|^2$ is an inviscid invariant and its flux to unresolved scales $< \ell$ is measured by

$$Z_\ell = -\boldsymbol{\nabla}\overline{\omega}_\ell\cdot\boldsymbol{\sigma}_\ell, \qquad (24)$$

where $\overline{\omega}_\ell$ is the filtered vorticity (perpendicular to the plane) and $\boldsymbol{\sigma}_\ell$ is the vorticity transport vector defined in (21). See [36, 37, 38]. From (24) one can see that enstrophy will cascade forward to small scales when vorticity transport tends to be "down-gradient" and $\boldsymbol{\nabla}\overline{\omega}_\ell\cdot\boldsymbol{\sigma}_\ell < 0$. On the other hand, enstrophy flux will be inverse to large-scales when the vorticity transport is "up-gradient." In $d = 2$ there are expected to be two inertial cascade ranges, the direct enstrophy cascade where the mean enstrophy flux is positive and the inverse energy cascade where the mean energy flux is negative [39, 40]. However, there is also some "leakage" of energy flux and enstrophy flux into the opposite ranges (e.g. see [36, 41]). In particular, the mean enstrophy flux in the inverse energy cascade range is *negative*, or toward larger scales. This means, according to (24), that the vorticity transport in that range is, on average, "up-gradient" or "anti-diffusive". Therefore, our argument for the sign of circulation-flux is reversed. In the inverse cascade range, a loop containing an excess of one sign of vorticity should tend to accumulate more vorticity of the same sign. Thus, in the $d = 2$ inverse energy cascade range there should be also an "inverse cascade of circulations" [72].

C. Stretching of Large-Scale Vorticity

We have seen that the "large-scale circulations", in the inertial range, evolve according to the equation

$$(d/dt)\overline{\Gamma}_\ell(C,t) = \oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell^*(t) \cdot d\mathbf{x}. \qquad (25)$$

The term on the righthand side due to the vortex-force need not be negligible. Thus, Taylor's conjecture that Kelvin's theorem should hold in the inertial range, even approximately, is far from obviously true. In the next section we shall explore this question mathematically, to the extent possible. Here we discuss some physical implications of Taylor's conjecture, if true.

If we suppose that the inertial-range circulations are conserved, then Taylor's argument about vortex line-stretching can be repeated for filtered vorticity, implying

$$(d/dt)\langle|\overline{\boldsymbol{\omega}}_\ell|^2\rangle > 0. \qquad (26)$$

This result can also be understood from the equation for the filtered vorticity, obtained by taking the curl of equation (15) (with $\nu = 0$):

$$\partial_t\overline{\boldsymbol{\omega}}_\ell + (\overline{\mathbf{u}}_\ell\cdot\boldsymbol{\nabla})\overline{\boldsymbol{\omega}}_\ell = (\overline{\boldsymbol{\omega}}_\ell\cdot\boldsymbol{\nabla})\overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\times\mathbf{f}_\ell^*. \qquad (27)$$

From this an equation for inertial-range enstrophy easily follows:

$$\partial_t\left(\frac{1}{2}|\overline{\boldsymbol{\omega}}_\ell|^2\right) + \boldsymbol{\nabla}\cdot\left[\frac{1}{2}|\overline{\boldsymbol{\omega}}_\ell|^2\overline{\mathbf{u}}_\ell + |\overline{\boldsymbol{\omega}}_\ell|\boldsymbol{\sigma}_\ell\right] \\ = \overline{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\omega}}_\ell + \mathbf{f}_\ell^*\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{\omega}}_\ell). \qquad (28)$$

[Compare with [42], eq.(51), for $\nu \to 0$.] Notice that the vorticity transport vector $\boldsymbol{\sigma}_\ell$ defined in (21) contributes to the space transport of enstrophy. However, assuming space-homogeneity, all of the space-flux terms average to zero and

$$(d/dt)\langle\frac{1}{2}|\overline{\boldsymbol{\omega}}_\ell|^2\rangle = \langle\overline{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\omega}}_\ell\rangle + \langle\mathbf{f}_\ell^*\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{\omega}}_\ell)\rangle. \qquad (29)$$

This equation is an exact inertial-range analogue of equation (8) for total enstrophy. The first term on the righthand side of (29) represents inertial-range vortex-stretching and the second term represents enstrophy flux to length-scales $< \ell$. For freely decaying turbulence at early times, Taylor's argument predicts that $\langle\overline{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\omega}}_\ell\rangle + \langle\mathbf{f}_\ell^*\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{\omega}}_\ell)\rangle = (d/dt)\langle\frac{1}{2}|\overline{\boldsymbol{\omega}}_\ell|^2\rangle > 0$. On physical grounds, one expects that the vortex-stretching is positive and the enstrophy transfer term negative, with the net enstrophy production positive. At later times a quasi-equilibrium should be established so that $(d/dt)\langle\frac{1}{2}|\overline{\boldsymbol{\omega}}_\ell|^2\rangle \approx 0$ and the dominant balance becomes

$$0 < \langle\overline{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\overline{\boldsymbol{\omega}}_\ell\rangle \approx -\langle\mathbf{f}_\ell^*\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{\omega}}_\ell)\rangle \qquad (30)$$

For some experimental results on these questions, see [42].

It was observed in [43] that the energy flux Π_ℓ to un-resolved scales $< \ell$ can be expressed approximately in terms of the negative skewness of filtered strain and the stretching rate of filtered vorticity:

$$\Pi_\ell = C\ell^2 \left[-\text{tr}\left(\overline{\mathbf{S}}_\ell^3\right) + (1/4)\overline{\boldsymbol{\varpi}}_\ell^\top \overline{\mathbf{S}}_\ell \overline{\boldsymbol{\varpi}}_\ell \right]. \qquad (31)$$

This expression is the first term in a systematic "multi-scale gradient expansion" [24]. It follows from an identity of Betchov [44] that for any homogeneous turbulence

$$\langle \Pi_\ell \rangle = C\ell^2 \langle \overline{\boldsymbol{\varpi}}_\ell^\top \overline{\mathbf{S}}_\ell \overline{\boldsymbol{\varpi}}_\ell \rangle. \qquad (32)$$

Thus, the energy cascade will be forward to small scales when the mean rate of vortex-stretching is positive. This is an inertial-range version of Taylor's mechanism [20, 21].

D. Anomalous Conservation of Circulation

We now consider the question whether Kelvin's Theorem can hold, in any sense, in turbulent flow at high Reynolds number. In view of equation (18) or (25), we must estimate the magnitude of the circulation-flux defined in (19). The following simple identity, observed in [17], is useful to provide an estimate of the turbulent subgrid force:

$$f_{\ell\, i}(\mathbf{x}) = \frac{1}{\ell} \int d\mathbf{r}\, (\partial_j G)_\ell(\mathbf{r})\, \delta u_i(\mathbf{r}; \mathbf{x}) \delta u_j(\mathbf{r}; \mathbf{x}) \qquad (33)$$
$$-\frac{1}{\ell} \int d\mathbf{r}\, (\partial_j G)_\ell(\mathbf{r})\, \delta u_i(\mathbf{r}; \mathbf{x}) \int d\mathbf{r}'\, G_\ell(\mathbf{r}')\, \delta u_j(\mathbf{r}'; \mathbf{x}).$$

Here $\delta\mathbf{u}(\mathbf{r}; \mathbf{x}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ is the velocity-increment with separation vector $\mathbf{r}$ at location $\mathbf{x}$. An upper bound easily follows that $|\mathbf{f}_\ell| = O(|\delta u(\ell)|^2/\ell)$, where $\delta u(\ell)$ is the maximum magnitude of the velocity-increment for separation vectors with $|\mathbf{r}| < \ell$ [17].

If the velocity field were smooth, then $|\delta u(\ell)| \sim$ (const.)ℓ for small ℓ and the subscale force would vanish as $\ell \to 0$. However, a turbulent velocity field does not remain smooth in the limit as the Reynolds number tends to infinity. Instead, theory, simulations, and experiment indicate that the velocity field is only Hölder continuous with exponent $0 < h < 1$:

$$|\delta\mathbf{u}(\mathbf{r}; \mathbf{x})| = O(r^h). \qquad (34)$$

At each point $\mathbf{x}$ one refers to the maximal value h for which (34) holds as the *Hölder exponent* at that point. There is a spectrum of such singularities in the flow, with exponent h occurring on a set $\mathcal{S}(h)$ with fractal dimension $D(h)$. It was pointed out by Onsager [9] that the smallest exponent $h_{\min}$ must be $\leq 1/3$ to explain non-vanishing energy dissipation in the inviscid limit. Parisi and Frisch [45] invoked a multifractal spectrum $D(h)$ of singularities to explain the anomalous scaling of pth moments of velocity-increments (so-called pth-order

structure-functions). Such multifractal spectra of Hölder exponents have been confirmed by analysis of data from experiments and simulations [14, 15, 16]. Of course, at finite Reynolds numbers there are only "near-singularities" in the inertial-range of scales and the velocity is smooth in the dissipation range, where effects of viscosity are important.

From our estimate below eq.(33), we see that $|\mathbf{f}_\ell| = O(\ell^{2h-1})$ at any point with local Hölder exponent h. Thus, the circulation flux $K_\ell(C, t)$ will go to zero as $\ell \to 0$ if the smallest velocity Hölder exponent $h_{\min}$ is $> 1/2$ and if also the curve $C(t)$ has finite length [17]. This is an exact analogue for circulation flux of Onsager's result [9] for vanishing of energy flux when $h_{\min} > 1/3$. Only a sufficiently rough velocity field can provide a transport of vortex lines which is non-vanishing in the limit as $\ell \to 0$. However, high Reynolds turbulence in space dimension $d = 3$ has a plethora of singularities with exponents $h \leq 1/2$. For example, the most probable exponent h_* with $D(h_*) = 3$ has a value $h_* \doteq 1/3$, very close to the mean-field Kolmogorov value [14, 15, 16]. Furthermore, the curves $\overline{C}_\ell(t)$ advected by the large-scale velocity $\overline{\mathbf{u}}_\ell$ are expected to approach a fractal curve $C(t)$ in the limit as $\ell \to 0$ [46, 47]. Thus, circulation-flux is not likely to vanish as the filtering length decreases through the inertial-range. Numerical simulations of high-Reynolds-number turbulence for $d = 3$ confirm this prediction [18].

There is an important subtlety in the formulation of Kelvin's theorem for infinite-Reynolds-number turbulence that must be mentioned at this point. Recent work on an idealized turbulence problem—the Kraichnan model of random advection [48]—has shown that Lagrangian particle trajectories $\mathbf{x}(t)$, $\mathbf{x}'(t)$ can explosively separate even when $\mathbf{x}_0 = \mathbf{x}'_0$ initially, if the advecting velocity field is only Hölder continuous and not Lipschitz. See [49]. Mathematically, this is a consequence of the non-uniqueness of solutions to the initial-value problem, while, physically, it corresponds to the two-particle turbulent diffusion of Richardson [50]. It has been rigorously proved in [51, 52] that there is a random process of Lagrangian particle paths $\mathbf{x}(t)$ in the Kraichnan model for a fixed realization of the advecting velocity and a fixed initial particle position. This phenomenon has been termed *spontaneous stochasticity* [53] and it is likely that it holds, not only in the Kraichnan model, but also for singular solutions of the inviscid Euler equations. If so, then the advected curves $C(t)$ that appear in the definition of circulation (2) are likely to be *random* fractal curves!

If these speculations are correct, then the time-series of circulations $\Gamma(C, t)$ are also a stochastic process, for a fixed turbulent velocity field. In [17] we have presented some plausibility arguments in favor of the following "martingale property" for this random process of circulations:

$$\langle \Gamma(C, t) | \Gamma(C, \tau), \tau < t' \rangle = \Gamma(C, t'), \text{ for } t > t'. \qquad (35)$$

Here $\langle \cdot \rangle$ denotes the expectation over the ensemble of random Lagrangian paths and we have conditioned on the

past circulation history $\{\Gamma(C,\tau),\tau<t'\}$. Heuristically,

$$(d/dt)\langle\Gamma(C,t)|\Gamma(C,\tau),\tau<t'\rangle =$$
$$-\lim_{\ell\to 0}\langle K_\ell(C,t)|\Gamma(C,\tau),\tau<t'\rangle = 0. \quad (36)$$

The circulation-flux in (36) is conjectured to average to zero, due to increasingly rapid oscillations of the vortex-force $\mathbf{f}_\ell^*$ around the loop $\overline{C}_\ell(t)$, as $\ell\to 0$. See [17]. The result in (36) has been partially confirmed by the results of a numerical simulation in [18], providing some support to the conjecture (35). This "martingale property" is a statement of conservation of circulations, in a conditional mean sense. It is not clear yet whether this weakened version of the Kelvin theorem is valid and, if so, whether it suffices for Taylor's vortex-stretching mechanism.

E.   Analogy with Phase-Slip in Superfluids

It is worth pointing out an analogy of the "circulation cascade" discussed above with another physical phenomenon, the "phase-slip" due to quantized vortex lines in superfluids [22, 54]. Anderson had already discussed classical analogues of quantum phase-slip in [22], Appendix B. His starting point was the classical Euler equations for an incompressible fluid, written as

$$\partial_t\mathbf{u} = -\boldsymbol{\nabla}h + \mathbf{u}\times\boldsymbol{\omega}, \quad (37)$$

where $h = p + (1/2)|\mathbf{u}|^2$ is the enthalpy. Anderson considered the line-integral of the fluid velocity $\mathbf{u}$ along a *stationary* curve C connecting two points P_1 and P_2, showing that

$$(d/dt)\int_C \mathbf{u}(t)\cdot d\mathbf{x} = -\Delta_C h + \int_C (d\mathbf{x}\times\mathbf{u})\cdot\boldsymbol{\omega}. \quad (38)$$

Here $\Delta_C h = h(P_2) - h(P_1)$ is the difference of h along the curve C. Denoting time-average by $\overline{(\cdot)}$, this relation yields

$$\Delta_C\overline{h} = \int_C \overline{(d\mathbf{x}\times\mathbf{u})\cdot\boldsymbol{\omega}}. \quad (39)$$

Since vortex lines for smooth solutions of the classical Euler equations move with the particle velocity $\mathbf{u} = d\mathbf{x}/dt$, the righthand side of (39) can be interpreted as an average rate of flow of vorticity across the curve C. This flow rate is thus equal to the average enthalpy difference along the curve. After deriving (39), Anderson wrote [22]: "We see immediately that this equation is far more important in a superfluid, where vorticity is conserved and quantized, than it is in ordinary fluids, where in a laminar flow, for instance, the right-hand side has little or no special significance." One critical difference between classical fluids and superfluids is that, in the former, the vortex-lines for laminar solutions move with the fluid. Thus, if one instead considers a *material* curve $C(t)$, advected by the fluid velocity $\mathbf{u}$, then one obtains

$$(d/dt)\int_{C(t)} \mathbf{u}(t)\cdot d\mathbf{x} = \Delta_{C(t)}\lambda \quad (40)$$

with $\lambda = (1/2)|\mathbf{u}|^2 - p$, rather than (38). The nontrivial term associated to flow of vorticity across the curve is now absent and eq.(40) for a closed loop yields the classical Kelvin Theorem.

Nevertheless, we have found that it is possible for turbulent flow to yield a nontrivial result. In fact, by filtering the Euler equation (37) one obtains

$$\partial_t\overline{\mathbf{u}}_\ell = -\boldsymbol{\nabla}\overline{h}_\ell + \overline{\mathbf{u}}_\ell\times\overline{\boldsymbol{\varpi}}_\ell + \mathbf{f}_\ell^*, \quad (41)$$

with the additional vortex-force term. This equation is equivalent to

$$\overline{D}_t\overline{\mathbf{u}}_\ell = \partial_t\overline{\mathbf{u}}_\ell + (\overline{\mathbf{u}}_\ell\cdot\boldsymbol{\nabla})\overline{\mathbf{u}}_\ell = -\boldsymbol{\nabla}\overline{p}_\ell^* + \mathbf{f}_\ell^*, \quad (42)$$

which is our old eq.(15) for $\nu = 0$. As we have seen in our earlier discussion of the large-scale circulation balance, eq.(18) or (25), the turbulent vortex-force provides a nontrivial transport of vorticity across material curves. Here it is crucial that the velocity field be sufficiently singular, to permit a transport which is non-vanishing for $\ell\to 0$. If instead the flow were smooth and laminar, then $\mathbf{f}_\ell^*\to 0$ in that limit and filtering the equation would lead to no new result. For singular solutions the Euler equation (37) *must* be filtered to make sense, as a matter of principle. In the presence of singularities the equation is interpreted in the sense of distributions, which means that it must be smeared with smooth test functions.

Nontrivial results are also possible in superfluids, for similar reasons. The superfluid phase order parameter φ obeys the Josephson-Anderson frequency equation [22, 54]:

$$\hbar\, d\varphi/dt = -(\mu + \frac{1}{2}mu_s^2), \quad (43)$$

where μ is the chemical potential and $\mathbf{u}_s = (\hbar/m)\boldsymbol{\nabla}\varphi$ is the superfluid velocity. It is straightforward to derive from (43) the superfluid equation of motion

$$D_t\mathbf{u}_s = (\partial_t + \mathbf{u}_s\cdot\boldsymbol{\nabla})\mathbf{u}_s = -\boldsymbol{\nabla}(\mu/m) - \mathbf{u}_s\times\boldsymbol{\omega}_s. \quad (44)$$

Here the final term contains the superfluid vorticity $\boldsymbol{\omega}_s = \boldsymbol{\nabla}\times\mathbf{u}_s$ which is, formally, a delta-function supported on singular vortex lines (zeroes of the superfluid density). Equation (44) is the basis of derivations of Kelvin's theorem for superfluids, e.g. see [55] in the context of the zero-temperature Gross-Pitaevskii equation. Note, however, that such derivations require that the advected loop not pass through singular points where the superfluid velocity is ill-defined. Since the quantized vortex lines are *not* material lines in general (e.g. see [56, 57, 58]), it is possible for them to migrate out of an advected loop. Examples are given in [55] of the failure of Kelvin's theorem due to the intersection of loops with singularities that are, formally, represented by the rightmost term in eq.(44). That equation is thus analogous to eq.(42) for classical turbulence.

One of the concrete manifestations of quantum phase slip is the decay of "persistent" superfluid flow in a thin

toroidal ring. E.g. see [59] and references therein. This process has a number of similarities to the "cascade of circulations" in turbulent flow. The decay of the super-flow is mediated by the (thermal or quantum) nucleation of quantized vortices which migrate out of the ring. The passage of a vortex across the toroidal cross-section induces by phase-slip a pulse of torque which decreases the circulation around the ring. The reduction in the angular momentum of the superfluid condensate is balanced by a gain in the normal fluid excitations, acting as an angular momentum reservoir. In the turbulent circulation-cascade, the large-scale vortex lines are also not material, because singularities in the velocity field allow them to diffuse relative to the fluid. The subscale modes at length-scales $< \ell$ act as a reservoir, whose feedback on the resolves scales $> \ell$ provides the vortex-force that drives the diffusion. Unlike in superfluids, this is a continuous process, since classical vortices are not quantized. There is also no need for the singularities to be nucleated as fluctuations, since they are everywhere present in the turbulent flow. Finally, if the "martingale" conjecture (35) is correct, then the turbulent diffusion of vortex-lines is not persistent in scale, on average, and does not lead to irreversible mean decay of circulations.

F. Scale-Locality and MSG Expansion

We have referred to this turbulent diffusion of vorticity as a "cascade" of circulations, but we have not shown that the process is a local-in-scale cascade. Here we shall examine this issue, following the general approach in [23].

We note first that the turbulent vortex-force $\mathbf{f}_\ell^*$ defined in (14) is *a priori* not ultraviolet (UV)-local, under conditions realistic for turbulence in $d = 3$. In fact, the vorticity is a dissipation-range variable and its largest contributions come from the viscous scale. The arguments in [23] for UV-locality would apply to $\mathbf{f}_\ell^*$ if the Hölder exponents h_u of velocity and h_ω of vorticity *both* were positive. However, $h_\omega = h_u - 1$, so that vorticity is expected to have negative Hölder exponents in the infinite-Reynolds-number limit (and thus to exist only as a distribution) [16]. It is possible that there could be cancellations in the average (14) over displacement vectors that defines the vortex-force. E.g. this was found to be true in the $d = 2$ enstrophy cascade, by an analysis of the results of a numerical simulation [38]. However, the UV-divergence is more severe for $d = 3$, so that sufficient cancellation is less likely there.

On the other hand, because of the identity (13), we may use the turbulent subscale force $\mathbf{f}_\ell = -\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_\ell$ rather than the vortex-force $\mathbf{f}_\ell^*$ to study the circulation-flux. The force $\mathbf{f}_\ell$ has much greater chance to be scale-local, because it is defined only in terms of velocity. Indeed, some locality properties follow directly from the representation (33) in terms of velocity increments. As in [23], let us define $\mathbf{u}^{>\Delta} = G_\Delta * \mathbf{u}$ to be the low-pass filtered velocity at length-scale $\Delta > \ell$ and define $\mathbf{u}^{<\delta} = \mathbf{u} - \mathbf{u}^{>\delta}$ to be

the high-pass filtered velocity at length-scale $\delta < \ell$. We can then define a very large-scale contribution $\mathbf{f}_\ell^{>\Delta}$ to the turbulent force by replacing both $\mathbf{u}$ in the formula (33) with $\mathbf{u}^{>\Delta}$. Likewise, we define a very small-scale contribution $\mathbf{f}_\ell^{<\delta}$ by replacing both $\mathbf{u}$ with $\mathbf{u}^{<\delta}$. Now suppose that the velocity field has Hölder exponent h at a considered point $\mathbf{x}$. Then, the following estimates can be easily derived, by the same methods as in [23]:

$$|\mathbf{f}_\ell^{>\Delta}| = O\left(\ell\Delta^{2h-2}\right) \tag{45}$$

and

$$|\mathbf{f}_\ell^{<\delta}| = O\left(\delta^{2h}/\ell\right). \tag{46}$$

The estimate (45) expresses infrared (IR)-locality. In fact, when $h < 1$, this estimate shows that $\mathbf{f}_\ell^{>\Delta}$ decreases for increasing Δ and fixed ℓ. Relative to the estimate $|\mathbf{f}_\ell| = O(\ell^{2h-1})$, the estimate (45) for $|\mathbf{f}_\ell^{>\Delta}|$ is smaller by a factor $O((\ell/\Delta)^{2(1-h)})$. Likewise, the estimate (46) expresses UV-locality. When $h > 0$, this estimate shows that $\mathbf{f}_\ell^{<\delta}$ decreases for decreasing δ and fixed ℓ. The estimate (46) for $|\mathbf{f}_\ell^{<\delta}|$ is smaller than that for $|\mathbf{f}_\ell|$ by a factor of $O((\delta/\ell)^{2h})$. These results show that most of the turbulent subgrid force $\mathbf{f}_\ell$ comes, pointwise, from pairs of velocity modes at length-scales $\sim \ell$.

The above arguments do not quite settle the issue of locality of the circulation-flux $K_\ell(C, t)$, however. The delicate point here is that large cancellations are expected in the line-integral of $\mathbf{f}_\ell$ that defines that flux. In order to infer scale-locality of $K_\ell(C, t)$, one must assume that similar cancellations occur in the line integrals of $\mathbf{f}_\ell^{>\Delta}$ and $\mathbf{f}_\ell^{<\delta}$. This issue is hard to address mathematically but may be investigated using data from simulation or experiment.

The UV-locality properties of the subgrid force $\mathbf{f}_\ell$ may be used to develop an analytical expression for it, by means of a multi-scale gradient expansion [24]. We consider only the lowest-order term in that expansion, which corresponds to the so-called "nonlinear model" for the stress [33]:

$$\tau_{ij} = C\ell^2 \overline{u}_{i,l}\overline{u}_{j,l}. \tag{47}$$

Here $C = \int d\mathbf{r}|G(\mathbf{r})|^2 r_1^2$ and a spherically-symmetric filter function is assumed, so that r_1 could be replaced with any other single component r_i. (In terms of the constant C_2 employed in [24], $C = C_2/d$ where d is the space dimension.) We use the convention of subscript ",j" to denote ∂_j, so that, for example, $u_{i,j} = \partial u_i/\partial x_j$. We also employ the Einstein summation convention for repeated indices. To avoid an excess of subscripts, we drop above and hereafter the subscript ℓ, since a fixed filter length will be always understood. The physical assumption behind the formula (47) is strong UV-locality, so that only adjacent subscale modes contribute to the stress. We expect that this extreme assumption is fairly good in the $d = 3$ energy cascade and the $d = 2$ direct enstrophy cascade. However, we present arguments below that it fails

badly for the $d = 2$ inverse energy cascade. Note that it is already known that the energy transfer is only weakly scale-local in $d = 2$ [60, 61, 62].

From the formula (47) for the stress, one obtains the corresponding formula for the subscale turbulent force:

$$f_i = -\partial_j \left(C\ell^2 \overline{u}_{i,l} \overline{u}_{j,l} \right). \qquad (48)$$

By means of a standard vector calculus identity, this can be written for $d \leq 3$ as:

$$f_i = C\ell^2 \epsilon_{ijk} \overline{u}_{j,l} \overline{\omega}_{k,l} - \partial_i \left(\frac{1}{2} C\ell^2 \overline{u}_{j,l} \overline{u}_{j,l} \right). \qquad (49)$$

Here ϵ_{ijk} is the anti-symmetric Levi-Civita tensor for $d = 3$. This formula can be simplified by substituting $\overline{u}_{j,l} = \overline{S}_{jl} - (1/2)\epsilon_{jlm}\overline{\omega}_m$ in the first term and $\overline{u}_{j,l}\overline{u}_{j,l} = \overline{S}_{j,l}\overline{S}_{j,l} + \frac{1}{2}|\overline{\omega}|^2$ in the second, yielding:

$$f_i = C\ell^2 \epsilon_{ijk} \overline{S}_{jl} \overline{\omega}_{k,l} - \partial_i \left(\frac{1}{2} C\ell^2 \overline{S}_{jl} \overline{S}_{jl} \right). \qquad (50)$$

This is our final formula for the turbulent force. Substituting (50) into (19) yields a similar formula for the circulation-flux:

$$K_\ell(C, t) = -C\ell^2 \oint_{\overline{C}_\ell(t)} \epsilon_{ijk} \overline{S}_{jl} \overline{\omega}_{k,l} \, dx_i. \qquad (51)$$

According to this formula, the diffusion of vortex-lines out of the loop is driven by strain acting upon the gradient of the vorticity vector. This is plausible, since the turbulent force should act to smooth out inhomogeneities in the large-scale vorticity field and become negligible when the latter is constant.

The same result (51) for the circulation-flux can be obtained from the "nonlinear model" of the turbulent vortex-force:

$$\begin{aligned} f_i^* &= C\ell^2 \epsilon_{ijk} \overline{u}_{j,l} \overline{\omega}_{k,l} \\ &= C\ell^2 \epsilon_{ijk} \overline{S}_{jl} \overline{\omega}_{k,l} + \partial_i \left(\frac{1}{4} C\ell^2 |\overline{\omega}|^2 \right). \end{aligned} \qquad (52)$$

Although this derivation yields the same result, it is theoretically less well-founded because of the poorer UV-locality properties of the vortex-force. On the other hand, it gives a little more physical insight, especially through the following alternative expression for the vortex-force:

$$\mathbf{f}^* = C\ell^2 \boldsymbol{\nabla}\cdot(\overline{\mathbf{S}}\times\overline{\boldsymbol{\omega}}) + \frac{1}{2} C\ell^2 (\overline{\boldsymbol{\omega}}\cdot\boldsymbol{\nabla})\overline{\boldsymbol{\omega}}. \qquad (53)$$

Here $(\overline{\mathbf{S}}\times\overline{\boldsymbol{\omega}})_{ji} = \epsilon_{ikl}\overline{S}_{jk}\overline{\omega}_l$ defines what was termed in [24] the "skew-strain matrix" for $d = 3$. Formula (53) is straightfowardly derived by calculating the divergence $(\overline{\mathbf{S}}\times\overline{\boldsymbol{\omega}})_{ji,j}$ and gathering the terms. This expression makes a nice connection with the MSG expansion for the turbulent stress, developed in [24]. The first term on the righthand side of (53) corresponds to one of the stress contributions in the MSG expansion, proportional

to "skew-strain". That term makes no strongly UV-local contribution to energy flux but a major contribution to helicity flux and here we see also to circulation flux.

The second term on the righthand side of (53) corresponds to another term from the MSG expansion in [24], a contractile stress along vortex-lines, $\tau_{ij}^{\text{vortex}} \propto -\overline{\omega}_i\overline{\omega}_j$. As discussed in [24], the effects of the small-scale turbulence give the large-scale vortex-lines "elastic" properties. The second term in (53) therefore has a simple geometric interpretation and can be written as

$$\begin{aligned} \mathbf{f}^{\text{vortex}} &= \frac{1}{2} C\ell^2 (\overline{\boldsymbol{\omega}}\cdot\boldsymbol{\nabla})\overline{\boldsymbol{\omega}} \\ &= \frac{\partial}{\partial s}\left(\frac{1}{4} C\ell^2 |\overline{\boldsymbol{\omega}}|^2 \right)\widehat{\boldsymbol{\omega}} + \frac{1}{2}\kappa C\ell^2 |\overline{\boldsymbol{\omega}}|^2 \widehat{\mathbf{n}}. \end{aligned} \qquad (54)$$

To derive (54) we have used the Frenet-Serret equations (e.g. see [63]) with $\widehat{\mathbf{t}} = \widehat{\boldsymbol{\omega}}$ the unit tangent vector along large-scale vortex lines, $\widehat{\mathbf{n}}$ the unit normal vector and $\widehat{\mathbf{b}}$ the binormal [73]. The term $\mathbf{f}_{\|}^{\text{vortex}}$ in (54) parallel to vortex lines arises from variations in the vortex-strength along the line. The term $\mathbf{f}_{\perp}^{\text{vortex}}$, which arises from bending of vortex lines, is proportional to the curvature κ of the line and is directed along the normal $\widehat{\mathbf{n}}$. Note that (54) gives a contribution to vorticity transport, $\boldsymbol{\sigma}^{\text{vortex}} = (1/2)\kappa C\ell^2 |\overline{\boldsymbol{\omega}}|^2 \widehat{\mathbf{b}}$, which is directed along the binormal, reminiscent of the velocity of a slender vortex filament in the local-induction approximation [25].

The formulas (50) and (52) for the turbulent force simplify in space dimension $d = 2$. In that case,

$$\begin{aligned} f_i &= -C\ell^2 \widetilde{\overline{S}}_{ij}(\partial_j\overline{\omega}) - \partial_i \left(C\ell^2 \overline{\sigma}^2 \right), \\ f_i^* &= -C\ell^2 \widetilde{\overline{S}}_{ij}(\partial_j\overline{\omega}) + \partial_i \left(\frac{1}{4} C\ell^2 |\overline{\omega}|^2 \right), \end{aligned} \qquad (55)$$

where $\pm\overline{\sigma}$ are the eigenvalues of the symmetric, traceless strain matrix $\overline{S}_{ij}$ and $\widetilde{\overline{S}}_{ij} = \overline{S}_{ik}\epsilon_{kj} = -\epsilon_{ik}\overline{S}_{kj}$ is another symmetric, traceless matrix, called in [61] the "skew-strain matrix" for $d = 2$. (Note that ϵ_{ij} is the $d = 2$ anti-symmetric Levi-Civita tensor.) The corresponding result for the circulation-flux is

$$K_\ell(C, t) = C\ell^2 \oint_{\overline{C}_\ell(t)} \widetilde{\overline{S}}_{ij}(\partial_j\overline{\omega}) \, dx_i. \qquad (56)$$

This result can be derived as well from equation (22) and the "nonlinear model" for the vorticity transport vector in $d = 2$,

$$\sigma_i = C\ell^2 \overline{u}_{i,j}(\partial_j\overline{\omega}), \qquad (57)$$

previously considered in [37, 38]. (This formula is equivalent to that for the vortex-force in eq.(55).) Note, however, that the formula (57) predicts "down-gradient" transport of vorticity whenever there is a positive rate of vorticity-gradient stretching and this is expected in $d = 2$ both for the direct enstrophy cascade [37, 38] and also the inverse energy cascade [61, 62]. "Down-gradient"

vorticity transport is qualitatively correct in the enstrophy cascade and there (56) may yield a good approximation. However, in the inverse energy cascade the vorticity transport must be "up-gradient" or "anti-diffusive." Therefore, (56) is not likely to be a good approximation in the inverse cascade range. It must be corrected by higher-order terms in the convergent MSG expansion, corresponding to smaller subgrid scales or higher-order gradients.

III. CONCLUSIONS

The main purpose of this paper was to elaborate a physical theory of "circulation cascade" in classical fluid turbulence. We have attempted to explain the conceptual basis of the phenomenon, its physical mechanisms, the scale-locality properties of the cascade, and its relation to inertial-range vortex-stretching and energy transfer. Clearly, there are many important issues that call for further work. Chief among these is to determine the validity of G. I. Taylor's proposed mechanism for turbulent energy-dissipation, based on vortex line-stretching [19, 20, 21]. Even after seventy years of research, basic elements of Taylor's proposal remain open to question. In particular, the strong inertial-range violations of the Kelvin Theorem—predicted in [17] and observed in [18]—cast some doubt on a key piece of Taylor's theory. It is possible that circulation conservation remains valid in some weaker sense, e.g. the conditional-mean version of the "martingale conjecture" in [17]. Further research is necessary to see whether any weaker form of the Kelvin Theorem holds at high Reynolds numbers and, if so, whether it is sufficient for the purposes of Taylor's mechanism. It should be emphasized that even the *existence* of circulations in the infinite-Reynolds-number limit is an open question. Advected loops in a turbulent flow are expected to become fractal [46, 47] and defining line-integrals for non-rectifiable curves demands some mathematical sophistication [17]. In superfluids the advected contours in Kelvin's Theorem can also become highly distorted, with interesting consequences for vortex motion [58]. Fractality of the advected loops could have significant implications for conservation of circulations in fluid turbulence.

In addition to hydrodynamics of incompressible fluids, there are other turbulent systems for which phenomena similar to "circulation cascade" are expected to exist. Of these, one of the most significant is magnetohydrodynamic (MHD) turbulence of plasmas. In this case it is Alfvén's Theorem [64] on conservation of magnetic flux in the ideal, zero-resistivity limit which plays the role of Kelvin's Theorem. However, there is strong evidence from observations of magnetic flux reconnection rates in astrophysical settings to believe that Alfvén's Theorem breaks down in MHD turbulence even with negligible resistivity [65, 66]. This violation of conservation of magnetic flux, presumably due to a similar cascade phenomenon as for Navier-Stokes dynamics, is discussed in a following paper for the MHD equations [67]. Another important problem is turbulence in geophysical fluids, where Ertel's Theorem [68] on conservation of potential vorticity (PV) plays a fundamental role in theories of quasi-geostrophy. It is well-known that Ertel's Theorem is a differential form of the Kelvin Theorem (e.g. see [69], Section 2.5 or [27], Section 4). The "cascade of circulations" in this context should be quite similar, generally speaking, to that for two-dimensional Navier-Stokes and correspond to a turbulent transport of PV out of the advected loop. However, in geophysical fluid dynamics there is an additional complication that the loop in Kelvin's theorem must lie in a surface of constant density (bouyancy) or pressure [27, 69]. Thus, turbulent mixing of isopycnal surfaces is an additional source of breakdown of Kelvin's Theorem in this context. Finally, another interesting setting for "circulation cascade" is superfluid turbulence [70]. The analogy between quantum phase-slip and circulation-cascade could prove useful here.

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[71] The circulation quantity defined in (17) is mainly of interest for $\ell < R$. Otherwise, if $\ell \gg R$, then $\overline{\mathbf{u}}_\ell$ is nearly constant over the scale of the loop and $|\overline{\Gamma}_\ell(C,t)| = O(R/\ell) \ll 1$.

[72] This phenomenon has been previously observed in numerical simulations of the $d = 2$ inverse energy cascade (Minping Wan and Shiyi Chen, private communication).

[73] Our notations here differ from those employed in section II B, where $\widehat{\mathbf{t}}_\ell$ referred to the unit tangent vector to the loop in the large-scale circulation and $\mathbf{n}_\ell = \widehat{\mathbf{t}}_\ell \times \widehat{\boldsymbol{\omega}}_\ell$.